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RESEARCH ARTICLE

# ON GRACEFUL LABELING OF CERTAIN SUBDIVISIONS OF SOME TRICYCLIC GRAPHS

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**ABSTRACT**

A labeled graph G which can be gracefully numbered is said to be graceful. Labeling the nodes of G with distinct nonnegative integers and then labeling the e edges of G with the absolute differences between node values, if the graph edge numbers run from 1 to e, the graph G is gracefully numbered. In this paper, we have discussed the gracefulness of some of the graphs formed from subdivisions of some tricyclic graphs.

**Key Words:** Labeling; Graceful graph; Hut graph.

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## INTRODUTION

Labeled graphs form useful models for a wide range of applications such as coding theory, X-ray crystallography, radar, astronomy, circuit design, communication network addressing and database management

A graceful labeling f of a graph G with q edges is an injective function from the vertices of G to the set {0,1,2,…,q} such that when each edge xy is assigned the label |f(x)-f(y)| , the resulting edge labels are distinct and nonzero. The concept above was put forward by Rosa in 1967.

The Ringel-Kotzig conjecture that all trees are graceful, has been the focus of many mathematicians. Among the trees known to be graceful are caterpillars, trees with at most 4-end vertices, trees with diameter at most 5, trees with at most 35 vertices, symmetrical trees, regular olive trees, lobsters, firecrackers, banana trees and bamboo trees.

Graphs consisting of any number of pairwise disjoint paths with common end vertices are called generalized theta graphs. Various labelings have been found for these graphs.

In this paper, some new classes of graphs have been constructed by combining some subdivisions of theta graphs with the star graphs St(n), ($n \geq 1$). Only finite simple undirected graphs are considered here. Our notations and terminology are as in [1]. We refer to [2] for some basic concepts.

**RESULTS ON GRACEFULNESS OF SOME SUBDIVISIONS OF THETA RELATED GRAPHS**

**Definition 2.1 [3]** The theta graph θ (α, β, γ) consists of three edge disjoint paths of lengths α, β, and γ having the same end points. Let the theta graph θ (2, 2, 3) have the paths $P_1$:$v_2$,$v_1$,$v_5$;$P_2$:$v_2$,$v_6$,$v_5$ and $P_3$:$v_2$,$v_3$,$v_4$,$v_5$.

**Definition 2.2** Attach an edge ($v_1$,$v_6$) to the theta graph θ (2, 2, 3) to form the hut graph θ' (2, 2, 3). This graph has three cycles.

**Definition 2.3 [4]** Let A be any graph and B be any tree graph. $A_i \circ B$ denotes the new graph formed by attaching a center vertex of B to a vertex $v_i$ of A. $A_i \,\square\, B$ denotes the new graph formed by attaching a center vertex of B to a vertex $v_i$ of A, by means of an edge.

**Definition 2.4 [5]** Let A be any graph. Let B and C be any tree graphs. $A_{ij} \circ (B,C)$ denotes the new graph formed by attaching a center vertex of B to a vertex $v_i$ of A and a center vertex of C to a vertex $v_j$ of A,where i and j are distinct. $A_{ij} \,\square\, (B,C)$ denotes the new graph formed by attaching a center vertex of B to a vertex $v_i$ of A by means of an edge and a center vertex of C to a vertex $v_j$ of A by means of an edge, where i and j are distinct.

**Theorem 2.5** $\theta'_{\alpha\beta}$ (2,2,3) □ (St(m),St(n)) is graceful, for m = n and m, n ≥ 1.

**Proof.** Consider the theta graph θ (2,2,3) with 6 vertices $v_1,v_2,\ldots,v_6$ .Add an edge $v_1v_6$ to this graph to form θ'(2,2,3). Let St(m) be a star graph with (m +1) vertices $w,w_1,w_2,\ldots,w_m$ for m ≥ 1,where w is the center vertex and $w_1,w_2,\ldots,w_m$ are pendant vertices.

Label the star graph St(n) on (n+1) vertices, by naming the vertices $u,u_1,u_2,\ldots,u_n$. ( n ≥ 1).Here u is the center vertex and the other vertices are pendant vertices. To form the graph G = $\theta'_{\alpha\beta}$ (2,2,3) □ (St(m),St(n)) attach the vertex w of St(m) with a vertex $v_\alpha$ ($1 \leq \alpha \leq 6$) of θ'(2,2,3) by an edge $wv_\alpha$ and attach the vertex u of St(n) with a vertex $v_\beta$ ($1 \leq \beta \leq 6$) of θ'(2,2,3) by an edge $wv_\beta$ ,for distinct α and β. G has (m + n + 8) vertices and (m + n + 10) edges. Let m and n be equal.

The vertex set V(G) = {u, w, $v_i$, $w_j$, $u_k$ / i = 1,2,…,6 , j = 1,2,…,m , k = 1,2,…,n }.

The edge set E(G) ={$v_i$ $v_{i+1}$ / i=1,2,…,5}∪{$v_i$ $v_{i+4}$ / i=1,2}∪{$ww_j$/j=1,2,…,m}∪{u $u_k$/ k=1,2,…,n}∪ {$v_i$ w, $v_j$ u / i, j =1or 2 or 3 or 4 or 5 or 6 and i ≠ j } ∪ {$v_i$ $v_{i+5}$ / i =1}.

Let f be the labeling on the set of vertices of G and g be the induced labeling on the set of edges of G.

The vertex label set of G can be written as A∪B∪C∪I where A={f($v_i$)/ i=1,2,…,6}; B = { f($w_j$) / j=1,2,…, m}; C = { f($u_k$) / k=1,2,…,n }; I = { f(u) , f(w)}.

The edge label set of G can be written as D∪K∪F∪L∪H∪J where D={g($v_i$ $v_{i+1}$) / i=1,2,…,5}; K={g($v_i$ $v_{i+4}$) / i=1,2}; F={g($ww_j$)/j=1,2,…,m};L ={g(u $u_k$) / k=1,2,…,n}; H ={g($v_i$ w), g($v_j$ u) / i, j =1or 2 or 3 or 4 or 5 or 6 and i ≠ j }; J ={g($v_i$ $v_{i+5}$) / i =1}. Consider all the cases upto isomorphism.

**Case 1** α = 1, β = 4

Let the labeling f on the vertices of G be defined by f($v_i$) = i - 1 for i = 1,3 ;

f($v_i$) = m + n + i + 5 for i = 2; f($v_i$) = m + i +2 for i = 4; f($v_i$) = m + n + i + 4 for i = 5; f($v_i$) = i – 5 for i = 6; f($w_j$) = j + 5 for 1 ≤ j ≤ m; f($u_k$) = m + k + 6 for 1 ≤ k ≤ n;

f(w) = m + n +10; f( u ) = m + n + 8.

The induced labeling g on the edges of G is defined by g($v_i$ $v_{i+1}$) = m+n+7 for i = 1; g($v_i$ $v_{i+1}$) = m+n+5 for i =2; g($v_i$ $v_{i+1}$) = n+4 for i = 3; g($v_i$ $v_{i+1}$) = n+3 for i = 4; g($v_i$ $v_{i+1}$) = m+n+8 for i = 5; g($v_i$ $v_{i+4}$) = m+n+9 for i =1; g($v_i$ $v_{i+4}$) = m+n+6 or i = 2; g(w $w_j$) = m+n+5 – j for 1≤j≤ m; g(u $u_k$) = n+2–k for 1≤ k≤ n; g(w $v_i$) = m+n+10 for i = 1; g(u $v_j$) = n +2 for j = 4; g($v_i$ $v_{i+5}$ ) = 1 for i = 1.

The vertex labels of G can be arranged in the following order.

A = {0,1,2,m +6, m+n+7, m+n+9}; B ={6,7,…,m+5}; C ={m+7,…,m+n+6}; I ={m+n +8, m+n+10}. The set of vertex labels of G is A∪B∪C∪I = {0,1,2, 6, … ,m+n+ 10}.

The edge labels of G can be arranged in the following order.

D ={n+3, n+4, m+n+5, m+n+7, m+n+8}; K ={m+n+6, m+n+9}; F ={n+5,…,m+n+4};

L ={2,…,n + 1}; H = {n + 2, m + n + 10 }; J = { 1 }. The set of edge labels of G isD ∪ K ∪ F∪ L∪ H∪J = {1, 2, 3, … , m + n +10 }.

Therefore the set of vertex labels and edge labels are distinct . So f is a graceful labeling. Hence G = $\theta'_{14}(2,2,3)$ □ (St(m),St(n)), for $m \geq 1$, $n \geq 1$, is a graceful graph.

**Case 2** α = 1, β = 5

Let the labeling f on the vertices of G be defined by $f(v_i) = i$ for i = 1; $f(v_i) = m + n + i + 6$ for i = 2; $f(v_i) = m + n + i + 5$ for i = 3; $f(v_i) = i – 4$ for i = 4; $f(v_i) = m + n + i + 4$ for i = 5; $f(v_i) = i – 2$ for i = 6; $f(w_j) = j + 5$ for $1 \leq j \leq m$; $f(u_k) = m + k + 6$ for $1 \leq k \leq n$; f(w) = 2; f( u ) = 3;

The induced labeling g on the edges of G is defined by $g(v_i v_{i+1}) = m+n+7$ for i = 1; $g(v_i v_{i+1}) = 2$ for i = 2; $g(v_i v_{i+1}) = m + n + 10$ for i = 3; $g(v_i v_{i+1}) = m + n + 9$ for i = 4; $g(v_i v_{i+1}) = m + n + 5$ for i = 5; $g(v_i v_{i+4}) = m + n + 8$ for i = 1; $g(v_i v_{i+4}) = m + n + 4$ for i = 2; $g(ww_j) = j + 3$ for $1 \leq j \leq m$; $g(uu_k) = m + k + 3$ for $1 \leq k \leq n$; $g(wv_i) = 1$ for i = 1;
$g(uv_j) = m + n + 6$ for j = 5;.$g(v_i v_{i+5}) = 3$ for i = 1. .

The vertex labels of G can be arranged in the following order.

A = { 0,1,4,m + n + 8, m + n + 9, m + n +10}; B ={6,7,…,m+ 5}; C = {m+7,…,m+n+6}, I = {2,3}. The set of vertex labels of G is A∪B∪C∪I = {0,1,2,3,4,6,…,m+5,m+7,…,m + n +6, m + n + 8, ...., m + n + 10}.

The edge labels of G can be arranged in the following order.

D ={2, m+ n+ 5, m+n+7, m+n+9, m+n+10}; K ={m+n+4, m+n+8}; F ={4,5,…,m+3};

L ={m+4,…,m+n+3}; H ={1,m+n+6}; J ={3}. The set of edge labels of G is D∪K∪F∪ L∪H∪J = {1, 2, 3, … , m + n +10 }.

Therefore the set of vertex labels and edge labels are distinct . So f is a graceful labeling. Hence G = $\theta'_{15}(2,2,3)$ □ (St(m),St(n)), for $m \geq 1$, $n \geq 1$, is a graceful graph.

**Case 3** α = 1, β = 6

Let the labeling f on the vertices of G be defined by $f(v_i) = i + 3$ for i = 1; $f(v_i) = i$ for i = 2; $f(v_i) = m + n + i + 4$ for i = 3; $f(v_i) = i – 4$ for i = 4; $f(v_i) = m + n + i + 5$ for i = 5; $f(v_i) = i – 5$ for i = 6; $f(w_j) = j + 4$ for $1 \leq j \leq m$; $f(u_k) = m + k + 5$ for $1 \leq k \leq n$; f(w) = m + n + 8; f( u ) =m + n + 9.

The induced labeling g on the edges of G is defined by $g(v_i v_{i+1}) = 2$ for i = 1; $g(v_i v_{i+1}) = m + n + 5$ for i = 2; $g(v_i v_{i+1}) = m + n + 7$ for i = 3; $g(v_i v_{i+1}) = m + n + 10$ for i = 4; $g(v_i v_{i+1}) = m + n + 9$ for i = 5; $g(v_i v_{i+4}) = m + n + 6$ for i = 1; $g(v_i v_{i+4}) = 1$ for i = 2; $g(ww_j) = m + n - j + 4$ for $1 \leq j \leq m$; $g(uu_k) = n – k + 4$ for $1 \leq k \leq n$; $g(wv_i) = m + n + 4$ for i = 1; $g(uv_j) = m + n + 8$ for j = 6; $g(v_i v_{i+5}) = 3$ for i = 1 .

The vertex labels of G can be arranged in the following order.

A = {0,1,2,4, m+n+7, m+n+10}; B ={5,6,7,…, m+ 4}; C = {m+6,…,m+n+5}; I = {m+n +8, m+n+9}.The set of vertex labels of G is A∪B∪C∪I = {0,1,2,4,…,m+4, m+6,…, m+ n+5, m+n+7,…, m+n+10}.

The edge labels of G can be arranged in the following order.

D = {2, m+n+5, m+n+7, m+n+9, m+n+10}; K ={1,m+n+6}, F={n+4,…,m+n+3};

L ={4,…, n+3}; H ={m+n+4, m+n+8}; J ={3}. The set of edge labels of G is D∪K∪F∪ L∪H∪J = {1,2,3,…,m+n +10 }.

Therefore the set of vertex labels and edge labels are distinct . So f is a graceful labeling. Hence G = $\theta'_{16}(2,2,3)$ □ (St(m),St(n)), for $m \geq 1$, $n \geq 1$, is a graceful graph.

**Case 4** α = 2, β = 4

Let the labeling f on the vertices of G be defined by $f(v_i) = i+1$ for i=1; $f(v_i) = i–2$ for i = 2; $f(v_i) = m+n+i+6$ for i = 3; $f(v_i) = i – 3$ for i = 4, 6; $f(v_i) = m+n+i+2$ for i = 5; $f(w_j) = n+j+6$ for $1≤ j ≤ m$; $f(u_k) = k+4$ for $1≤ k ≤ n$; f(w) = m + n + 10; f( u ) = m + n + 8.

The induced labeling g on the edges of G is defined by $g(v_i v_{i+1}) = 2$ for i = 1; $g(v_i v_{i+1}) = m + n + 9$ for i = 2; $g(v_i v_{i+1}) = m + n + 8$ for i = 3; $g(v_i v_{i+1}) = m + n + 6$ for i = 4; $g(v_i v_{i+1}) = m + n + 4$ for i = 5; $g(v_i v_{i+4}) = m + n + 5$ for i = 1; $g(v_i v_{i+4}) = 3$ for i = 2; $g(ww_j) = m + 4 – j$ for $1≤ j ≤ m$; $g(uu_k) = m + n + 4 – k$ for $1 ≤ k ≤ n$; $g(wv_i) = m + n + 10$ for i = 2; $g(uv_j) = m + n + 7$ for j = 4; $g(v_i v_{i+5}) = 1$ for i = 1.

The vertex labels of G can be arranged in the following order.

A = {0, 1, 2, 3,,m + n + 7, m + n + 9}; B = {n + 7,…, m + n + 6}; C = {5,6,7,…,n + 4}, I = {m+n+8, m+n+10}.The set of vertex labels of G is A∪B∪C∪I ={0,1,2,3,5,….,n + 4, n + 7, … , m + n + 10}.

The edge labels of G can be arranged in the following order.

D = {2, m+n+4, m+n+6, m+n+8, m+n +9 }; K ={3, m+n+5}; F ={4,…,m+3}, L={m+4, m + n + 3 }, H = { m + n + 7, m + n + 10 }, J = { 1 }. The set of edge labels of G is D ∪ K∪ F∪ L∪ H∪J = {1, 2, 3, … , m + n +10 }.

Therefore the set of vertex labels and edge labels are distinct . So f is a graceful labeling. Hence G = $θ_{24}(2,2,3)$ □ (St(m),St(n)), for $m ≥ 1$, $n ≥ 1$, is a graceful graph.

**Case 5** α = 2, β = 5

Let the labeling f on the vertices of G be defined by $f(v_i) = i – 1$ for i = 1; $f(v_i) = m + n + i + 8$ for i = 2; $f(v_i) = n + i + 3$ for i = 3; $f(v_i) = i – 4$ for i = 4; $f(v_i) = m + n + i + 3$ for i = 5; $f(v_i) = i – 5$ for i = 6; $f(w_j) = j + 5$ for $1 ≤ j ≤ m$; $f(u_k) = m + k + 6$ for $1 ≤ k ≤ n – 1$; $f(u_k) = m + k + 9$ for k = n; f(w) = m + 7; f( u ) = 3.

The induced labeling g on the edges of G is defined by $g(v_i v_{i+1}) = m + n + 10$ for i = 1; $g(v_i v_{i+1}) = m+4$ for i = 2; $g(v_i v_{i+1}) = m+2$ for i = 3; $g(v_i v_{i+1}) = m+n+4$ for i = 4; $g(v_i v_{i+1}) = m+n+7$ for i = 5; $g(v_i v_{i+4}) = m+n+8$ for i = 1;$g(v_i v_{i+4}) = m+n+9$ for i = 2; $g(ww_j) = m+2–j$ for $1 ≤ j ≤ m$; $g(uu_k) = m+k+3$ for $1≤ k ≤ n – 1$; $g(uu_k) = m+k+6$ for k=n; $g(wv_i) = m +3$ for i = 2; $g(uv_j) = m + n + 5$ for j = 5; $g(v_i v_{i+5}) = 1$ for i = 1.

The vertex labels of G can be arranged in the following order.

A = { 0,1,4, m+6, m+n+8, m+n+10}; B ={6,7,…, m+5}; C={m+8,…, m+n+6, m+n+9}; I ={3,m+7}.The set of vertex labels of G is A∪B∪C∪I ={0,1,3,4,6,…,m+5,m+6,…,m+ n + 6, m + n + 8, …, m + n + 10}.

The edge labels of G can be arranged in the following order.

D ={m+2, m+4, m+n+4, m+n+7, m+n+10}; K ={m+n+8, m+n+9}; F = {2,3,…,m+1};

L ={m+5,…,m+n+3, m+n+6}; H ={m+3, m+n+5}; J ={1}. The set of edge labels of G is D ∪ K ∪ F∪ L∪ H∪J = {1, 2, 3, … , m + n +10 }.

Therefore the set of vertex labels and edge labels are distinct . So f is a graceful labeling. Hence G = $θ'_{25}(2,2,3)$ □ (St(m),St(n)), for $m ≥ 1$, $n ≥ 1$, is a graceful graph.

**Case 6** α = 3, β = 4

Let the labeling f on the vertices of G be defined by $f(v_i) = i$ for i = 1; $f(v_i) = i+2$ for i = 2; $f(v_i) = m+n + i + 5$ for i = 3; $f(v_i) = m + n + i + 6$ for i = 4; $f(v_i) = i – 5$ for i = 5;
$f(v_i) = m + n + i + 3$ for i = 6; $f(w_j) = j + 5$ for $1 ≤ j ≤ m$; $f(u_k) =m + k + 6$ for $1 ≤ k ≤ n$; f(w) = 2; f( u ) = 3.

The induced labeling g on the edges of G is defined by $g(v_i v_{i+1}) = 3$ for i = 1; $g(v_i v_{i+1}) = m + n + 4$ for i = 2; $g(v_i v_{i+1}) = 2$ for i = 3; $g(v_i v_{i+1}) = m + n + 10$ for i = 4; $g(v_i v_{i+1}) = m + n + 9$ for i = 5; $g(v_i v_{i+4}) = 1$ for i = 1; $g(v_i v_{i+4}) = m + n + 5$ for i = 2; $g(ww_j) = j + 3$ for $1 ≤ j ≤ m$; $g(uu_k) = m + k + 3$ for $1 ≤ k ≤ n$; $g(wv_i) = m+n + 6$ for i = 3; $g(uv_j) = m + n + 7$ for j = 4; $g(v_i v_{i+5}) = m + n + 8$ for i = 1.

The vertex labels of G can be arranged in the following order.

A = {0,1,4,m+n+8,m+n+9,m+n+10}; B ={6,7,…,m+5}; C = {m+7,…,m+n+6};

I ={2,3}.The set of vertex labels of G is A∪B∪C∪I = {0,1,2,3,4,6,…,m+5, m+7,…, m+ n + 6, m + n + 8, … , m + n +10}.

The edge labels of G can be arranged in the following order.

D ={2,3,m+n + 4, m+n+9, m+n+10}; K = {1, m+n+5}; F ={4,…, m+3}; L = {m+4, … , m + n + 3}; H = { m + n + 6, m + n + 7}}; J = { m + n + 8 }. The set of edge labels of G is D ∪ K ∪ F∪ L∪ H∪J = {1, 2, 3, … , m + n +10 }.

Therefore the set of vertex labels and edge labels are distinct . So f is a graceful labeling. Hence G = $\theta'_{34}(2,2,3)$ □ (St(m),St(n)), for *m* ≥ 1, *n* ≥ 1, is a graceful graph.

**Case 7** α = 4, β = 5

Let the labeling f on the vertices of G be defined by $f(v_i)$ = i – 1 for i = 1; $f(v_i)$ = m+n+i+8 for i = 2; $f(v_i)$ = i+2 for i = 3; $f(v_i)$ = m+n+i+4 for i = 4,5; $f(v_i)$ = i – 4 for i = 6; $f(w_j)$ =n+j+6 for 1 ≤ j ≤ m; $f(u_k)$ = k+5 for 1≤ k ≤ n; f(w) = 4; f( u ) = 3.

The induced labeling g on the edges of G is defined by $g(v_i v_{i+1})$ = m+n+10 for i = 1; $g(v_i v_{i+1})$ = m+n+5 for i = 2; $g(v_i v_{i+1})$ = m + n + 3 for i = 3; $g(v_i v_{i+1})$ = 1 for i = 4; $g(v_i v_{i+1})$ = m + n + 7 for i = 5; $g(v_i v_{i+4})$ = m + n + 9 for i = 1; $g(v_i v_{i+4})$ = m + n + 8 for i = 2; $g(ww_j)$ = n + j + 2 for 1 ≤ j ≤ m; $g(uu_k)$ = k + 2 for 1 ≤ k ≤ n; g(wvi) = m + n + 4 for i = 4; g(uvj) =m + n + 6 for j = `5; $g(v_i v_{i+5})$ = 2 for i = 1.

The vertex labels of G can be arranged in the following order.

A = {0,2,5, m+n+8, m+n+9, m+n+10}; B = {n+7,…,m+n+6}; C ={6,…, n+5}, I= {3,4}. The set of vertex labels of G is A∪B∪C∪I = {0, 2,…, n + 5, n + 7, … , m + n +6, m + n + 8, … ,m + n + 10}.

The edge labels of G can be arranged in the following order.

D = {1, m + n + 3, m + n + 5, m + n + 7, m + n + 10 }; K = {m + n + 8, m + n + 9}; F = {n + 3, …, m + n + 2 }; L = {3, 4,…, n + 2 }; H={m + n + 4, m + n + 6 }}; J = { 2 }. The set of edge labels of G is D ∪ K ∪ F∪ L∪ H∪J = {1, 2, 3, … , m + n +10 }.

Therefore the set of vertex labels and edge labels are distinct . So f is a graceful labeling. Hence G = $\theta'_{45}(2,2,3)$ □ (St(m),St(n)), for *m* ≥ 1, *n* ≥ 1, is a graceful graph.

**Case 8** α = 4, β = 6

Let the labeling f on the vertices of G be defined by $f(v_i)$ = i – 1 for i = 1;

$f(v_i)$ = m+n+ i+5 for i = 2,5; $f(v_i)$ = i for i = 3; $f(v_i)$ = i + 1 for i = 4; $f(v_i)$ = i – 5 for i = 6; $f(w_j)$ = j+ 5 for 1 ≤ j ≤ m; $f(u_k)$ = m+k+6 for 1≤ k ≤ n; f(w) = m+n+8; f( u ) = m + n + 9.

The induced labeling g on the edges of G is defined by $g(v_i v_{i+1})$ = m + n + 7 for i = 1; $g(v_i v_{i+1})$ = m + n + 4 for i = 2; $g(v_i v_{i+1})$ = 2 for i = 3; $g(v_i v_{i+1})$ = m + n + 5 for i = 4; $g(v_i v_{i+1})$ = m + n + 9 for i = 5; $g(v_i v_{i+4})$ = m + n + 10 for i = 1; $g(v_i v_{i+4})$ = m + n + 6 for i = 2; $g(ww_j)$ = m + n + 3 – j for 1 ≤ j ≤ m; $g(uu_k)$ = n + 3 – k for 1 ≤ k ≤ n; $g(wv_i)$ = m + n + 3 for i = 4; $g(uv_j)$ = m + n + 8, for j = 6; $g(v_i v_{i+5})$ = 1 for i = 1 .

The vertex labels of G can be arranged in the following order.

A ={0,1,3,5, m+n+7, m+n+10}; B ={6,7,…, m+5}; C ={m+7,…, m+n+6}; I ={m+n+8, m+n+9}.The set of vertex labels of G is A ∪ B ∪ C ∪ I = {0,1,3,5,…, m+5, m+7,…, m+n +10}.

The edge labels of G can be arranged in the following order.

D = {2, m + n + 4, m + n + 5, m + n + 7, m + n + 9 }; K = {m + n + 6, m + n + 10 }; F = {n +3, ..., m+ n + 2}; L = {3, ... , n + 2 }; H ={m+ n + 3, m + n + 8 }; J = { 1 }. The set of edge labels of G is D ∪ K ∪ F∪ L∪ H∪J = {1, 2, 3, … , m + n +10 }.

Therefore the set of vertex labels and edge labels are distinct . So f is a graceful labeling. Hence G = $\theta'_{46}(2,2,3)$ □ (St(m),St(n)), for *m* ≥ 1, *n* ≥ 1, is a graceful graph.

**Case 9** α = 5, β = 6

Let the labeling f on the vertices of G be defined by $f(v_i)$ = i for i = 1; $f(v_i)$ = m + n + i + 5 for i = 2; $f(v_i)$ = m + n + i +6, for i = 3; $f(v_i)$ = i + 1 for i = 4; $f(v_i)$ = m+n + i + 5 for i = 5; $f(v_i)$ = i – 6 for i = 6; $f(w_j)$ = j + 5 for 1 ≤ j ≤ m; $f(u_k)$ = m + k + 6 for 1 ≤ k ≤ n; f(w) = 2; f( u ) = 3.

The induced labeling g on the edges of G is defined by $g(v_i v_{i+1})$ = m+n+6 for i=1; $g(v_i v_{i+1})$ = 2 for i = 2; $g(v_i v_{i+1})$ = m + n + 4 for i = 3; $g(v_i v_{i+1})$ = m + n + 5 for i = 4; $g(v_i v_{i+1})$ = m + n + 10 for i = 5; $g(v_i v_{i+4})$ = m + n + 9 for i = 1;

$g(v_i v_{i+4})$ = m + n + 7 for i = 2; g $g(ww_j)$ = j + 3 for 1 ≤ j ≤ m; $g(uu_k)$ = m + k + 3 for 1 ≤ k ≤ n; $g(wv_i)$ = m+n + 8 for i = `5; $g(uv_j)$ = 3 for j = 6; $g(v_i v_{i+5})$ = 1 for i = 1 .

The vertex labels of G can be arranged in the following order.

A={0,1,5, m+n+7, m+n+9, m+n+10}; B={6,7,…, m+5}; C ={m+7,…,m+n+6}; I ={2,3}. The set of vertex labels of G is A∪B∪C∪I = {0,1,2,3,5,6,…,m+5,m+7, …, m + n +7, m + n + 9, m + n + 10}.

The edge labels of G can be arranged in the following order. D = {2, m + n + 4, m + n + 5, m + n + 6, m + n + 10 }, K = {m + n + 7, m + n + 9}, F = {4, 5, … , m + 3 }, L = { m + 4, … , m + n + 3 }, H = { 3, m + n + 8}, J = { 1 }.. The set of edge labels of G is D ∪ K ∪ F∪ L∪ H∪J = {1, 2, 3, … , m + n +10 }.

Therefore the set of vertex labels and edge labels are distinct . So f is a graceful labeling. Hence G = $\theta'_{56}(2,2,3)$ □ (St(m),St(n)), for $m \geq 1$, $n \geq 1$, is a graceful graph.

**REFERENCES**

[1] Harary, F., Graph Theory, Addision Wesley , New Delhi,1972.

[2] Gallian, J.A., A dynamic survey of graph labeling, The Electronic Journal of Combinatorics, 19(2012), # DS6.

[3]. Raziya Begam, R., Palanivelrajan,M., Gunasekaran, K., and Shahul Hameed, A.R., Graceful labeling of some theta related graphs, International. Journal of Fuzzy Mathematical Archive, 2(2013), 78-84

[4]. Shahul Hameed, A.R., Palanivelrajan, M., Gunasekaran, K., and Raziya Begam, R., On Graceful labeling of some Bicyclic graphs, International. Journal of Fuzzy Mathematical Archive, 3(2013), 1-8.

[5] Shahul Hameed, A.R., Palanivelrajan, M., Gunasekaran, K., and Raziya Begam, R., Graceful labeling of some combinations of hut graphs, Archimedes J. Math**.,** 4(1)(2014)**,** 27-35.