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NOTES ON FRAMES OF SUBSPACES

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ABSTRACT

The aim of this article is to give details of research progress made during the last few years. This article is devoted to the study of frames of subspaces, their properties and applications. The notion of frames of subspaces is a recent development that provides a natural framework for two-stage data processing.

Key words and phrases: Frames, Frames of Subspaces, weights.

INTRODUCTION

Frames are generalizations of orthonormal bases in Hilbert spaces. As for an orthonormal basis, a frame allows each element in the underlying Hilbert space to be written as an unconditionally convergent infinite linear combination of the frame elements; however in contrast to the situation for a basis, the coefficients might not be unique. General frame theory of subspaces (or fusion frames) was introduced by P. Casazza and G.Kutyniok [3] and M. Fornasier [5] as a natural generalization of frame theory in Hilbert spaces. Fornasier uses subspaces which are quasi-orthogonal to construct local frames and piece them together to get global frames. P. Casazza and G.Kutyniok formulate a general method for piecing together local frames to get global frames.

2. FRAMES

Definition 2.1[2]: A sequence $\{f_i\}_{i\in I}$ of elements of a Hilbert space H is called a *frame* if there exist constants $A, B > 0$ such that for all $f \in H, A\|f\|^2 \leq \sum_{i\in I}|\langle f, f_i\rangle|^2 \leq B\|f\|^2$.

The numbers A and B are called the lower and upper frame bounds respectively. The largest number $A > 0$ and the smallest number $B > 0$ satisfying the frame inequalities for all $f \in H$ are called optimal frame bounds.

Definition 2.2[2]: Let $\{e_n\}$ be an orthonormal basis for an infinite dimensional Hilbert space H and $f_n \in H$ for all $n \in \mathbb{Z}$, then the operator $T: H \to H$ given by $Te_n = f_n$ is called the preframe operator or *synthesis operator* associated with$\{f_n\}$.

Definition 2.3[2]: The operator T^*defined by $T^*f = \sum_n \langle f, f_n\rangle\, e_n$ for all $f \in H$ is called the *analysis operator*.

Definition 2.4[2]: Operator $S = TT^*$ is an invertible operator on H called the *frame operator*.

3. FRAMES OF SUBSPACES

Let H be a separable Hilbert space. All subspaces are assumed to be closed. Countable (or finite) sets I, J_idenote index sets. Let $P_W$denote the orthogonal projection of H onto W.

Definition 3.1[3]: Let $\{v_i\}_{i\in I}$ be a family of weights. A family of closed subspaces $\{W_i\}_{i\in I}$ of a Hilbert space $H$is a *frame of subspaces* with respect to $\{v_i\}_{i\in I}$for H, if there exist constants $0 < C \leq D < \infty$ such that for all $f \in H, C\|f\|^2 \leq \sum_{i\in I} {v_i}^2 \left\|P_{W_i}(f)\right\|^2 \leq D\|f\|^2$.

Definition 3.2[3]: The family $\{W_i\}_{i\in I}$ is called a C-tight frame of subspaces with respect to $\{v_i\}_{i\in I}$, if in the definition 3.1, the constants C and D can be chosen such that $C = D$, and a *Parseval frame of subspaces* with respect to $\{v_i\}_{i\in I}$ if $C = D = 1$.

We will study the conditions under which weight sequences for frame of subspaces for H give rise to frames for H and vice versa.

Theorem 3.3[3]: Let for each $i \in I$,

i. $v_i > 0$

ii. $\{f_{ij}\}_{j\in J_i}$ be a frame sequence in H with frame bounds A_i and B_i.

iii. $W_i = \overline{span}_{j\in J_i}\{f_{ij}\}and\{e_{ij}\}_{j\in J_i}$ be an orthonormal basis for subspaceW_i.

Let A and B be such that$0 < A = \inf A_i \leq B = \sup B_i < \infty$. Then the following conditions are equivalent:

a. $\{v_i f_{ij}\}_{i\in I, j\in J_i}$ is a frame for H.

b. $\{v_i e_{ij}\}_{i\in I, j\in J_i}$ is a frame for H.

c. $\{W_i\}_{i\in I}$is a frame of subspaces with respect to $\{v_i\}_{i\in I}$ for H.

Definition 3.4[3]: A family of subspaces $\{W_i\}_{i\in I}$of H is said to be complete, if $\overline{span}_{i\in I}\{W_i\} = H$.

Lemma 3.5[3]: Let $\{W_i\}_{i\in I}$ be a family of subspaces in H, and let $\{v_i\}_{i\in I}$ be a family of weights.

If $\{W_i\}_{i\in I}$ is a frame of subspaces with respect to $\{v_i\}_{i\in I}$ for H, then it is complete.

Lemma 3.6[3]: Let $\{W_i\}_{i\in I}$ be a family of subspaces in H. Let for each $i \in I, \{e_{ij}\}_{j\in J_i}$ be an orthonormal basis for $\{W_i\}$. Then $\{W_i\}_{i\in I}$ is complete if and only if $\{e_{ij}\}_{j\in J_i}$ is complete.

Theorem 3.7[3]: The removal of a subspace from a frame of subspaces with respect to some family of weights leaves either a frame of subspaces with respect to the same family of weights or an incomplete family of subspaces.

Proof: Let $\{W_i\}_{i\in I}$ be a frame of subspaces with respect to $\{v_i\}_{i\in I}$ for H. Let for each $i \in I, \{e_{ij}\}_{j\in J_i}$ be an orthonormal basis for $\{W_i\}$. By theorem 3.3, $\{v_i e_{ij}\}_{i\in I, j\in J_i}$ is a frame for H. Let $i_0 \in I$, by theorem 5.4.7[4], $\{v_i e_{ij}\}_{i\in I, j\in J_i}$ is either a frame or an incomplete set. If it is a frame, then again by theorem 3.3, $\{W_i\}_{i\in I\setminus\{i_0\}}$ is a frame of subspaces with respect to $\{v_i\}_{i\in I}$ for H. If $\{v_i e_{ij}\}_{i\in I\setminus\{i_0\}, j\in J_i}$ is an incomplete set then $\{e_{ij}\}_{i\in I\setminus\{i_0\}, j\in J_i}$ is an incomplete set and hence by lemma 3.6, $\{W_i\}_{i\in I\setminus\{i_0\}}$ is incomplete.

The following lemma shows that the intersection of the elements of a frame of subspaces with a subspace still leaves a frame of subspaces for a smaller space.

Lemma 3.8[3]: Let V be a subspace of H and let $\{W_i\}_{i\in I}$ be a frame of subspaces with respect to $\{v_i\}_{i\in I}$ for H with frame bounds C and D.Then $\{W_i \cap V\}_{i\in I}$ is a frame of subspaces with respect to $\{v_i\}_{i\in I}$ for V with frame bounds C and D.

Proof: Let $f \in V \subseteq H$. Then $C\|f\|^2 \le \sum_{i\in I} v_i^2 \left\|P_{W_i}(f)\right\|^2 \le D\|f\|^2$. Also since $f \in V$ we have

$$\sum_{i\in I} v_i^2 \left\|P_{W_i}(f)\right\|^2 = \sum_{i\in I} v_i^2 \left\|P_{W_i}P_V(f)\right\|^2 = \sum_{i\in I} v_i^2 \left\|P_{W_i\cap V}(f)\right\|^2$$

Thus $C\|f\|^2 \le \sum_{i\in I} v_i^2 \left\|P_{W_i\cap V}(f)\right\|^2 \le D\|f\|^2$. This implies that $\{W_i \cap V\}_{i\in I}$ is a frame of subspaces with respect to $\{v_i\}_{i\in I}$ for V with frame bounds C and D.

4. SYNTHESIS AND ANALYSIS OPERATOR

In this section Synthesis and Analysis operators are defined for frame of subspaces and the properties of these operators are discussed.

Definition 4.1[1]: Let $\{W_i\}_{i\in I}$ be a family of subspaces in H. The space $(\sum_{i\in I} \oplus W_i)_{l_2}$ is defined as $(\sum_{i\in I} \oplus W_i)_{l_2} = \{\{f_i\}_{i\in I} : f_i \in W_i \text{ and } \sum_{i\in I}\|f_i\|^2 < \infty\}$.

Definition 4.2[1]: Let $\{W_i\}_{i\in I}$ be a frame of subspaces with respect to $\{v_i\}_{i\in I}$ for H. Then the *synthesis operator* for $\{W_i\}_{i\in I}$ and $\{v_i\}_{i\in I}$ is the operator $T_{W,v} : (\sum_{i\in I} \oplus W_i)_{l_2} \to H$ defined by

$$T_{W,v}(f) = \sum_{i\in I} v_i f_i \text{ for all } f = \{f_i\}_{i\in I} \in (\sum_{i\in I} \oplus W_i)_{l_2}.$$

Definition 4.3[1]: The adjoint $T^*_{W,v}$ of the synthesis operator is called the *analysis operator*.

Theorem 4.4[3]: Let $\{W_i\}_{i\in I}$ be a frame of subspaces with respect to $\{v_i\}_{i\in I}$ for H. Then the analysis operator $T^*_{W,v}: H \to (\sum_{i\in I} \oplus W_i)_{l_2}$ is given by $T^*_{W,v}(f) = \{v_i P_{W_i}(f)\}_{i\in I}$.

The relations between frame of subspaces and the associated analysis and synthesis operators can be seen in the following theorem.

Theorem 4.4[1]: Let $\{W_i\}_{i\in I}$ be a family of subspaces in H, and let $\{v_i\}_{i\in I}$ be a family of weights. Then the following conditions are equivalent:

a. $\{W_i\}_{i\in I}$ is a frame of subspaces with respect to $\{v_i\}_{i\in I}$ for H.
b. The synthesis operator $T_{W,v}$ is bounded, linear and onto.
c. The analysis operator $T^*_{W,v}$ is an (possibly onto) isomorphism.

Proof: Let $f \in H$.Then$\|T^*_{W,v}(f)\|^2 = \left\|\{v_i P_{W_i}(f)\}_{i\in I}\right\|^2 = \sum_{i\in I} v_i{}^2\|P_{W_i}(f)\|^2$. Let $\{W_i\}_{i\in I}$ be a frame of subspaces with respect to $\{v_i\}_{i\in I}$ for H with frame bounds C and D. Then for each $f \in H$, $C\|f\|^2 \le \sum_{i\in I} v_i{}^2\|P_{W_i}(f)\|^2 \le D\|f\|^2$ which implies

$C\|f\|^2 \le \|T^*_{W,v}(f)\|^2 \le D\|f\|^2$.

Equivalently,$\sqrt{C}\|f\| \le \|T^*_{W,v}(f)\| \le \sqrt{D}\|f\|$. Therefore $T^*_{W,v}$is an isomorphism. Conversely, let$T^*_{W,v}$ be an isomorphism. Then there exists a constant$M > 0$ such that for each $f \in H$,

$$\frac{1}{M}\|f\| \le \|T^*_{W,v}(f)\| \le M\|f\|.$$

Squaring throughout, we get $\frac{1}{M^2}\|f\|^2 \le \|T^*_{W,v}(f)\|^2 \le M^2\|f\|^2$.

Therefore, $\frac{1}{M^2}\|f\|^2 \le \sum_{i\in I} v_i{}^2\|P_{W_i}(f)\|^2 \le M^2\|f\|^2$. This implies that $\{W_i\}_{i\in I}$ is a frame of subspaces with respect to $\{v_i\}_{i\in I}$ for H. Hence (a) and (c) are equivalent. The equivalence of (b) and (c) holds for each operator on a Hilbert space H.

5. FRAME OPERATOR

Casazza and Kutyniok in [3] defined associated frame operator for each frame of subspaces.

Definition 5.1 [3]: Let $\{W_i\}_{i\in I}$ be a frame of subspaces with respect to $\{v_i\}_{i\in I}$ for H.Then the *frame operator*$S_{W,v}$ for $\{W_i\}_{i\in I}$ and $\{v_i\}_{i\in I}$ is defined by

$$S_{W,v}(f) = T_{W,v}T^*_{W,v}(f) = \sum_{i\in I} v_i{}^2 P_{W_i}(f).$$

Theorem 5.2 [3]: Let $\{W_i\}_{i\in I}$ be a frame of subspaces with respect to $\{v_i\}_{i\in I}$ with frame bounds C and D. Then the frame operator $S_{W,v}$ for $\{W_i\}_{i\in I}$ and $\{v_i\}_{i\in I}$ is a positive, self-adjoint, invertible operator on H with $CI_H \le S_{W,v} \le DI_H$ where I_H is the identity operator on H. We further have reconstruction formula $f = T_{S^{-1}_{W,v}}T^*_{W,v}(f) = \sum_{i\in I} v_i{}^2 S^{-1}_{W,v}P_{W_i}(f)$ $for\ all\ f \in H$.

Proof: Let $f \in H$. Since$\{W_i\}_{i\in I}$ is a frame of subspaces with respect to $\{v_i\}_{i\in I}$ with frame bounds C and D, we have $C\|f\|^2 \le \sum_{i\in I} v_i{}^2\|P_{W_i}(f)\|^2 \le D\|f\|^2$.

Now,$\langle S_{W,v}(f), f\rangle = \langle \sum_{i\in I} v_i{}^2 P_{W_i}(f), f\rangle = \sum_{i\in I} v_i{}^2\langle P_{W_i}(f), f\rangle = \sum_{i\in I} v_i{}^2\|P_{W_i}(f)\|^2 \ge 0$.

This implies that $S_{W,v}$ is a positive operator and hence is self- adjoint. Now,

$$\langle Cf, f\rangle = C\|f\|^2 \le \sum_{i\in I} v_i{}^2\|P_{W_i}(f)\|^2 = \langle S_{W,v}(f), f\rangle \le D\|f\|^2 = \langle Df, f\rangle.$$

This implies that $CI_H \le S_{W,v} \le DI_H$. $S_{W,v}$is invertible as $\|CI_H - S_{W,v}\| < 1$. Again,

$$f = S^{-1}_{W,v}S_{W,v}(f) = S^{-1}_{W,v}\sum_{i\in I} v_i{}^2 P_{W_i}(f) = \sum_{i\in I} v_i{}^2 S^{-1}_{W,v}P_{W_i}(f)$$

and, $T_{S^{-1}_{W,v}}T^*_{W,v}(f) = T_{S^{-1}_{W,v}}\left(\{v_i P_{W_i}(f)\}_{i\in I}\right) = \sum_{i\in I} v_i{}^2 S^{-1}_{W,v}P_{W_i}(f) = S^{-1}_{W,v}S_{W,v}(f) = f$.

Example 5.3: Let $\{e_i\}_{i\in\mathbb{Z}}$ be an orthonormal basis for some Hilbert space H. Define the subspaces W_1, W_2 by $W_1 = \overline{span}_{i\geq 0}\{e_i\}$and $W_2 = \overline{span}_{i\leq 0}\{e_i\}$. Then $\{W_1, W_2\}$ is a frame of subspaces with respect to weights $\{v_1, v_2\}$ with $v_1 = v_2 = v$, since

$$v_1\left\|P_{W_1}(f)\right\|^2 + v_2\left\|P_{W_2}(f)\right\|^2 = \sum_{i\in\mathbb{Z}} v|\langle f, e_i\rangle|^2 + v|\langle f, e_0\rangle|^2 = v\|f\|^2 + v|\langle f, e_0\rangle|^2$$

and

$$v\|f\|^2 \leq v\|f\|^2 + v|\langle f, e_0\rangle|^2 \leq 2v\|f\|^2$$

Therefore,

$$v\|f\|^2 \leq \sum_{i=1}^{2} v_i\left\|P_{W_i}(f)\right\|^2 \leq 2v\|f\|^2.$$

Theorem 5.4 [3]: Let $\{W_i\}_{i\in I}$ be a family of subspaces in H, and let $\{v_i\}_{i\in I}$ be a family of weights. Then the following conditions are equivalent:

a. $\{W_i\}_{i\in I}$ is a Parseval frame of subspaces with respect to $\{v_i\}_{i\in I}$ for H.

b. $S_{W,v} = I_H$.

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