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RESEARCH ARTICLE

ANTI-HAUSDORFF U-SPACES

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ABSTRACT

This is the third in a series of papers on U-spaces. Here Anti-Hausdorffness has been introduced for U- spaces and many topological theorems related to anti-Hausdorffness have been generalized to U- spaces, as an extension of study of supratopological spaces.

Key Words: U-space, Trivial anti-Hausdorff U-space, Non-trivial anti-Hausdorff U-space, U- Continuous image, Quotient U-space, Irreducible.

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INTRODUCTION

In a previous paper [1] we have introduced U- spaces and studied some of their properties. In this paper we use the terminology of [1]. Some study of these spaces was done previously in ([2],[3],[4],[5]) in less general form, and the spaces were called supratopological spaces. Anti-Hausdorff topological space was introduced and studied in [6]. In this paper the concept of anti-Hausdorff U-spaces has been introduced and a few important properties of such spaces have been studied. A number of interesting examples have been constructed to prove non- trivialness of such results.

2. ANTI- HAUSDORFF U- SPACES

We have generalized some results on anti-Hausdorff topological spaces in [6] to U-spaces. We recall that a U- space X is a non-empty set X together with a collection U of subsets of X such that U is closed under unions and X and Φ belong to U . A U-space is called trivial if it is a topological space.

Definition 2.1 [6] A U-space X with $|X|\geq 2$ is said to be anti-Hausdorff U-space, if for every pair of distinct points x, y in X and pair of distinct U-open sets G and H such that $x\in G$, $y\in H$, $G\cap H\neq\Phi$, i.e., if no two distinct points can be separated by disjoint U-open sets.

Here, $|X|$ denoted the number of elements of X. An anti-Hausdorff U-space which is not a topological space will be called a non-trivial anti-Hausdorff U-space. Otherwise it is called trivial. It is easily seen that an anti-Hausdorff U-space X is non-trivial only if $|X|\geq 3$.

Example 2.1 Let X = {a, b, c}, U_1 = {X, Φ, {a, b},{a, c}} and U_2 = {X, Φ,{b, c},{a, c}}. Then (X, U_1) and (X, U_2) are non-trivial anti-Hausdorff U-spaces.

Example 2.2 Let X = {a, b, c, d} and U_1 = {X, Φ, {a, b, c},{a, d}},
U_2 = {X, Φ,{a, b},{a, c},{a, b, c}}. Then (X, U_1) and (X, U_2) are non-trivial anti-Hausdorff U-spaces.

Example 2.3 Let X = N, U = {X, Φ, {1, 2, 3},{1, 4, 5}, {1, 2, 3, 4, 5}}.
Then (X, U) is a non-trivial anti-Hausdorff U-space.

Example 2.4 Let X = R, U = {X, Φ, N, Z, 2 Z, N $\cup$ 2Z }.
Then (X, U) is a non-trivial anti-Hausdorff U-space.

Theorem 2.1 [6] A U-subspace of a non-trivial anti-Hausdorff U-space need not be anti-Hausdorff.
Proof: Let us consider the U-space (X, U), where X = {a, b, c, d} and U = {X, Φ,{a, b},{a, c}, {a, b, c}}.
Then (X, U) is a non-trivial anti-Hausdorff U-space, since there is no pair of disjoint non-empty U-open sets in X. Now let Y = {b, c}.
Then as a subspace of X, Y has the U-structure, U = {Y, Φ, {b},{c}}.
Obviously, Y is not anti-Hausdorff U-space.

Theorem 2.2 [6] If A and B are two non-trivial anti-Hausdorff U-subspaces of a U-space X, then the subspace A $\cap$ B need not be a non-trivial anti-Hausdorff U-space.
Proof: Let X = {a, b, c, d, e, f }, U = {X, Φ,{a, b, c},{ b, c, d},{a, b, c, d}, {b, c, d, e, f}}. Clearly (X, U) is a non-trivial U-space. Let A = {a, c, d, f } and B = {a, b, d, f}. Then A and B are U-subspace of X with
U_A = {A, Φ, {a, c},{c, d}, {a, c, d}, {c, d, f}}, U_B = {B, Φ,{a, b},{b, d}, {a, b, d}, {b, d, f}}.
Clearly both A and B are non-trivial anti-Hausdorff U-subspaces of X.
Now A $\cap$ B = {a, d, f } and $U_{A \cap B}$ ={A $\cap$ B, Φ, {a}, {d},{a, d},{d, f}}.
Then A $\cap$ B is a trivial U-space, which is not anti-Hausdorff.

In the situation of Theorem-2.2, it is also possible that A $\cap$ B is a non-trivial anti-Hausdorff U-space as is shown by the following example.
Example 2.5 Let X= {a, b, c, d, e}, U = {X, Φ, {a, b},{a, b, c},{a, c, d, e}}. Clearly (X, U) is a non-trivial U-space. Let A = {a, b, c, d} and B = {a, b, c, e}. Then A and B are U-subspaces of X with U_A = {A, Φ, {a, b}, {a, b, c},{a, c, d}}, U_B = {B, Φ, {a, b},{a ,b, c},{a, c, e}}.
Clearly both A and B are non-trivial anti-Hausdorff U-subspaces.
Now A $\cap$ B = {a, b, c} and $U_{A \cap B}$ = {A $\cap$ B, Φ,{a, b}, {a, c}} which is a non- trivial anti-Hausdorff U-space.

Remark 2.1 [6] If A_1 and A_2 are two non-trivial subspaces of a non-trivial U-space X, then the subspace $A_1 \cap A_2$ may be non-trivial anti-Hausdorff U-space even if neither A_1 nor A_2 is so.
Proof: Let X = {a, b, c, d, f }, U = { X, Φ,{a}, {b, c},{c, d}, {a, b, c}, {a, c, d}, {f}, {b, c, f}, {c, d, f }, {a, f }, {a, b, c, f }, {a, c, d, f }, {b, c, d}, {a, b, c, d},{b, c, d, f}}.
Clearly, X is a non-trivial U-space.
Let A_1 = {a, b, c, d} and A_2 = {b, c, d, f }.

Then the U-structure U_{A1} and U_{A2} on A_1 and A_2 respectively are U_{A1} = {A_1, Φ, {a},{b, c,{c, d}, {a, b, c},{a, c, d},{b, c, d}} and U_{A2} = {A_2, Φ, {f},{b, c}, {c, d},{b, c, f}, {b, c, d}, {c, d, f }}.

Clearly both A_1 and A_2 are non-trivial subspaces of a U-space X, neither of which is anti-Hausdorff.

Now $A_1 \cap A_2$ = {b, c, d} and $U_{A1 \cap A2}$ ={$A_1 \cap A_2$, Φ, {b, c}, {c, d}}.
Thus $A_1 \cap A_2$ is a non- trivial anti- Hausdorff U-space.

Theorem 2.3 [6] Let A_1 and A_2 be two anti-Hausdorff U-spaces with U-structures U^1 and U^2 respectively. Then $(A_1 \cup A_2, \langle U^1 \cup U^2 \rangle)$ need not be anti-Hausdorff U-space. Here $\langle U^1 \cup U^2 \rangle$ is the U-structure generated by $U^1 \cup U^2$ in $A_1 \cup A_2$.

Proof: Let A_1 = {a, c, d, e} U^1 = {A_1, Φ, {a},{a, c}, {a, c, d},{a, d, e}}, and A_2 = {b, c, d, e} U^2 = {A_2, Φ, {b},{b, c},{b, c, d},{b, d, e}}. Then (A_1, U^1) and (A_2, U^2) are non-trivial anti-Hausdorff U-spaces.

Then A = $A_1 \cup A_2$ = {a, b, c, d, e}. Let U be the U-structure on A generated by $U^1 \cup U^2$, i.e., U ={A, A_1, A_2, Φ,{a},{b},{a, c},{b, c},{a, c, d}, {a, d, e},{b, c, d}, {b, d, e},{a, b},{a, b, c},{a, b, c, d},{a, b, d, e}}. So, in (X, U), a $\in$ {a}, b $\in$ {b} with {a},{b} $\in$ U and {a} $\cap$ {b} = Φ.
Hence (X, U) is not an anti- Hausdorff U-space.

Theorem 2.4 [6] Every U-continuous image of an anti- Hausdorff U-space is an anti-Hausdorff U-space.
Proof: Let X, Y be two U-spaces where X is anti-Hausdorff U-space. Let f be a U-continuous map of X onto Y. Let y_1 and y_2 be two distinct points of Y, and let H_1 and H_2 be two U-open sets in Y such that $y_1 \in H_1$, $y_2 \in H_2$. Since f is onto there exist x_1, x_2 in X such that $f(x_1) = y_1$, $f(x_2) = y_2$. Let $G_1 = f^{-1}(H_1)$, $G_2 = f^{-1}(H_2)$. Since f is U-continuous, both G_1 and G_2 are U-open sets. Since X is anti-Hausdorff U-space, $G_1 \cap G_2 \neq \Phi$. Let $x \in G_1 \cap G_2$, then $f(x) \in H_1 \cap H_2$. Thus $H_1 \cap H_2 \neq \Phi$. So, Y is anti-Hausdorff U-space.

Definition 2.2 Let (X, U) be U-space and R an equivalence relation on X. The equivalence class for each $x \in X$ is denoted by $\bar{x}$. We define U-structure $\overline{\mathcal{U}}$ on the collection of equivalence classes $\frac{X}{R}$ of X with respect to R as follows. Any subset $\bar{V}$ of $\frac{X}{R}$ will be a member of $\overline{\mathcal{U}}$ iff $\{x \in X | \bar{x} \in \bar{V}\} \in$ U, i.e., the collection of equivalence classes of every U-open set V of X is
U-open in $\frac{X}{R}$ and these are the only U-open members of $\frac{X}{R}$.

This U-structure $\overline{\mathcal{U}}$ is called the identification U-structure or the quotient U-structure on X, and $(\frac{X}{R}, \overline{\mathcal{U}})$ is called the identification U-space or the quotient U-space of X with respect to R.

Corollary 2.1. If X is an anti-Hausdorff U-space and R is an equivalence relation on X, then the quotient U-space $\frac{X}{R}$ is anti-Hausdorff U-space.

Proof: It follows from the definition of quotient U-space that the map $f : X \rightarrow \frac{X}{R}$ given by f(x) = cls x is continuous and onto. The proof is then follows from Theorem 2.4.

Definition 2.3 [6] A U-space X is said to be U-irreducible if every pair of non-empty U-open sets in X intersect. Thus a U-space X is U-irreducible if, for every pair of non-empty U-open sets V, W in X, $V \cap W \neq \Phi$.

Theorem 2.5 [6] Let X be a U-space. For the statements:
(i) X is anti-Hausdorff U-space,
(ii) X is U-irreducible,
(iii) Every non- empty U-open set in X is connected U-space,
(iv) Every non- empty U-open set in X is dense in X,
following implications hold: (i) $\Leftrightarrow$ (ii), (iii) $\Rightarrow$ (ii) and (ii) $\Leftrightarrow$ (iv).
Proof: We first prove (i) $\Leftrightarrow$ (ii).
To prove (i) $\Rightarrow$ (ii) let X be a anti- Hausdorff U-space. If possible suppose that X is not
U-irreducible. Then there exist non- empty U-open sets V and W in X such that $V \cap W = \Phi$. Since V and W are non- empty, there exist $x \in V$ and $y \in W$. Since $V \cap W = \Phi$, $x \neq y$. X being anti-Hausdorff U-space, this is a contradiction. Therefore X is U-irreducible.

We now prove (ii) $\Rightarrow$ (i). Let X be U-irreducible. If possible, let X be not anti-Hausdorff U-space. Then there exist x, y $\in$ X with x $\neq$ y and U-open sets V and W in X with V $\cap$ W = Φ and x $\in$ V, y $\in$ W. Since V and W are non-empty, this is a contradiction to the fact that X is U-irreducible.
Hence X is anti-Hausdorff U-space.

To prove (iii) $\Rightarrow$ (ii), let every U-open set in X be connected U-space. If X is not U-irreducible, then there exist non-empty U-open sets V_1 and V_2 in X, such that $V_1 \cap V_2 = \Phi$. This implies that the U-open set $V = V_1 \cup V_2$ is a disconnected U-open set in X. This is a contradiction to our hypothesis. Hence X is U-irreducible.

We now prove (ii) $\Leftrightarrow$ (iv). Let X be a U-irreducible space. Let V be a non- empty U-open set in X and let x $\in$ X. Let W be a U-open set in X such that x $\in$ W. Then W $\neq$ Φ. Since X is U-irreducible, V $\cap$ W $\neq$ Φ. So, x $\in \bar{V}$. Thus X = $\bar{V}$. Thus (ii) $\Rightarrow$ (iv).

Conversely, suppose every non-empty U-open set in X is dense in X. Let V and W be two non-empty U-open sets in X and let x $\in$ V. Since $\bar{W}$ = X and V is a neighborhood of x, V $\cap$ W $\neq$ Φ. So X is U-irreducible.
Therefore (iv) $\Rightarrow$ (ii). The proof of the theorem is thus complete.

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